Modification of quantum many-body relaxation by perturbations exhibiting a banded matrix structure

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We investigate how the observable relaxation behavior of an isolated quantum many-body system is modified in response to weak-to-moderate perturbations within a nonperturbative typicality framework. A key role is played by the so-called perturbation profile, which characterizes the dependence of the perturbation matrix elements in the eigenbasis of the unperturbed Hamiltonian on the difference of the corresponding energy eigenvalues. In particular, a banded matrix structure is quantitatively captured by a perturbation profile, which approaches zero for large energy differences. The temporal modification of the relaxation is linked to the perturbation profile via a nonlinear integral equation, which admits approximate analytical solutions for sufficiently weak and strong perturbations, and for which we work out a numerical solution scheme in the general case. As an example, we consider a spin lattice model with a pronounced banded matrix structure, and we find very good agreement of the numerics with our analytical predictions without any free fit parameter.

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I. INTRODUCTION

Despite their microscopic chaoticity [1,2], the macroscopically observable behavior of isolated quantum many-body systems is often surprisingly regular. For instance, it is by now well established that these systems generically equilibrate and usually even thermalize [2–4] and that the approach to equilibrium quite often follows a rather simple and direct route. Understanding how this dynamics emerges from a microscopic description, however, is still a theoretical challenge that has recently attracted considerable attention. A particularly interesting question in this context is how the observable relaxation behavior of a given system is modified under the influence of reasonably weak perturbations, linking, for example, analytically tractable simple systems (e.g., noninteracting, integrable) to generic ones (e.g., interacting, nonintegrable).

Characterizing the response of a given system to a perturbation is a recurrent problem in many areas of physics. Arguably the standard approach is to expand the pertinent equations of motion in terms of the perturbation strength and to solve the resulting hierarchy of simplified equations iteratively. Unfortunately, such a strategy is doomed to failure in the case of quantum systems with many degrees of freedom. Because of their extremely dense energy spectra, the concomitant small denominators of a perturbative expansion limit its applicability to extremely short timescales much shorter than the observed relaxation times. While there is strong evidence that related concepts like Fermi’s golden rule and linear response theory can describe many-body dynamics in certain scenarios [5–7], this somewhat surprisingly holds despite the many-body character and not because of it.

Here we tackle the question of how a many-body system responds to perturbations by “nonperturbative” methods, namely a typicality approach that aims to extract and separate the macroscopically relevant perturbation characteristics from the huge number of microscopic degrees of freedom. Our starting point is an isolated many-body quantum system described by a time-independent reference Hamiltonian $H$, and prepared in some initial state far from equilibrium. Provided that we know the observable relaxation dynamics of this unperturbed reference system, we ask how the behavior is changed when adding a weak-to-moderate perturbation $\lambda V$. In other words, the system still starts from the same initial state but now evolves in time according to the perturbed Hamiltonian

$$H_\lambda := H + \lambda V .$$  \hspace{1cm} (1)

One situation that could be modeled by such an approach is an unperturbed system composed of two isolated subsystems at equilibrium, which are then coupled sufficiently weakly via the perturbation $V$ and relax to a new, joint equilibrium state. Another interesting scenario arises when the reference system $H$ is integrable, in which case one can often calculate the unperturbed behavior analytically. In particular, integrable systems usually still equilibrate (just like the nonintegrable ones), meaning that expectation values of experimentally relevant observables approach a constant value and stay there for most of all later times. However, these integrable systems (unlike the nonintegrable ones) may not thermalize; i.e., equilibrium expectation values are not described by the pertinent thermodynamic equilibrium ensemble and call for extensions like generalized Gibbs ensembles instead [8–11]. Adding a
small integrability-breaking perturbation commonly leads to
“prethermalization” [4,6,12–15], meaning that the system still
follows the unperturbed (nonthermalizing) behavior for quite
some time before eventually departing toward the associated
thermal state. As a third example, more generally, one may
think of the unperturbed system as some system for which
the relaxation dynamics happens to be known, and ask for
the behavior when changing some parameter of the Hamiltonian
e.g., a “quantum quench” [3,11,16].

Basing our analysis on previous results from Ref. [17],
we recap those findings in Secs. II and III. More precisely,
in Sec. III. A crucial role is played by the resolvent (an
ounced theoretical prediction of the many-body response
the key assumptions of our theory in Sec. II, and establish the
we introduce the considered classes of systems and formulate
explicit and to compare it to numerical examples for random-
matrix and spin models. Finally, we summarize and discuss
our results in Sec. VI.

II. SCOPE AND PREREQUISITES

Before presenting our main result, we introduce the setting
and collect several key assumptions about the physical
situations we aim to describe (see also Supplemental Material of
Ref. [17] for further technical details).
The isolated many-body quantum system of reference
is described by a time-independent Hamiltonian
is prepared in some (pure or mixed, and
generally far from equilibrium) initial state with density operator
ρ(0). According to textbook quantum mechanics, the state at
any later time is then given by ρ(t) = e−iHt ρ(0)eit (h = 1). Of
primary interest to us are the time-dependent expectation values
(A)_ρ(0):= Tr[ρ(t) A] of self-adjoint operators A which
model some experimentally or theoretically relevant observ-
able, such as (sums of) local and/or few-body operators [2–4].
Similarly, the time-evolved state of the perturbed system with
Hamiltonian H_p from (1) is given by ρ_p(t) = e−iH_p t ρ(0)eit.

Overall, the main objective of our present work is to
establish quantitative predictions for the perturbed dynamics
(A)_ρ(t) based on the unperturbed behavior (A)_ρ(0) and some
essential characteristics of the perturbation V.

Regarding the systems under study, the following four key
assumptions will be taken for grant hereafter:

(i) The system should exhibit a well-defined macroscopic
energy, implying that the initial state ρ(0) [and hence also ρ(t)
at any later t] only significantly populates levels E_ν ∈ I within
a macroscopically small energy window I := [E, E + Δ]. In
particular, it is assumed that the density of states (DOS)

\[ D(E) := \sum_{\nu : E_\nu \in I} \delta(E - E_\nu) \]  

is approximately constant throughout I, \( D(E) \approx \varepsilon^{-1} \) with the
mean level spacing \( \varepsilon \). At the same time, the system’s many-
boson character entails that the window I is still micro-
copically large in the sense that the number of levels contained in I
is exponentially large in the system’s degrees of freedom
[18]. We emphasize that the initial state does not necessarily
define the window I. On the contrary, the window I can be
to some extent an arbitrary interval with the two prerequisites
that it exhibits an approximately constant DOS and contains all
E_ν with non-negligible level populations \( \langle \nu | \rho(0) | \nu \rangle \).

(ii) The perturbation should be sufficiently weak so as to
leave the thermodynamic properties of the system basically
unchanged. Notably, phase transitions induced by the pertur-

ation are thus ruled out. Recalling from textbook statistical
mechanics that the DOS in (2) is related to the Boltzmann entropy
\( S(E) \) via \( D(E) = e^{S(E)}/h^{S(E)} \) with Boltzmann’s constant
\( k_B \), this assumption particularly implies that also the
DOS of the perturbed \( H_p \) remains approximately constant with
mean level spacing \( \varepsilon \) cf., assumption (i). Because of the
generic level repulsion of interacting many-body systems [1],
the spectrum of \( H_p \) is typically indeed rather stiff, meaning
that the individual eigenvalues exhibit very fast fluctuations
upon variation of \( \lambda \), while their density only changes very
slowly [19].

We remark that such negligible changes of the thermo-
dynamic properties do not rule out interesting and nontrivial
changes of the relaxation dynamics, notably if the unperturbed
Hamiltonian is in some sense special (e.g., integrable, com-
muting with A or \( \rho(0) \), etc.); see also the examples below
Eq. (1) and in Sec. V.

(iii) The perturbation should be sufficiently strong so that
it significantly mixes a large number of unperturbed levels.
Denoting by \( | m \rangle \), the eigenvectors of \( H_p \) in (1), this is to say
that the overlaps

\[ U_{m\nu} := \langle m | \nu \rangle \]  
in between the unperturbed and perturbed eigenvectors should
extend across a scale \( \Gamma \) with \( \Gamma \gg \varepsilon \); i.e., \( U_{m\nu} \) should be
non-negligible (in a coarse-grained sense, see below) as long as
\( |E_m - E_\nu| \lesssim \Gamma \) [see also Eq. (7)]. On the other hand, note
that assumptions (i) and (ii) practically require \( \varepsilon \ll \Delta [20],
where \( \Delta \) is the width of the energy window I from (2). The extreme
density of levels of typical many-body systems [see
below Eq. (2)] still leaves room for a large range of parameters
\( \lambda \) such that \( \varepsilon \ll \Delta \). In particular, we can and will take for

ganted that the number of levels \( \mathcal{N}_v := \Gamma/\varepsilon \) that get mixed
by the perturbation is still exponentially large in the system’s
degrees of freedom \( f [17], \) i.e.,

\[ \mathcal{N}_v := \Gamma/\varepsilon = 10^{O(f)} \]  

Without going into the details, we remark that perturbations
which do not satisfy the requirement \( \varepsilon \ll \Gamma \) turn out (as
one might have expected) to actually be so weak that they
do not notably modify the unperturbed relaxation on any
reasonable timescale. Incidentally, the same behavior will also
be correctly reproduced by our final results. In this sense, the
requirement \( \varepsilon \ll \Gamma \) is not really indispensable.

So far, these considerations have been very general and did
not exploit any more specific properties of the actual system at
hand. To make any progress, it is clear that some information
about the perturbation \( V \) and possibly also the observable
A and initial state \( \rho(0) \) must be taken into account. The
common lore of statistical physics furthermore suggests that,
de spite its microscopic complexity, the observable behavior
of a many-body system can usually be described in terms of
a relatively small number of macroscopic (coarse-grained)
quantities, for instance some appropriately defined (local) densities. This brings us to our main assumption about the structure of admissible perturbations:

(iv) On a coarse-grained level, the magnitude of the perturbation matrix elements \( V_{\mu \nu} := \langle \mu | V | \nu \rangle \) within the energy window \( I \) should only depend on the energy difference \( |E_{\mu} - E_{\nu}| \) of the coupled levels \([21]\), i.e.,

\[
|\langle V_{\mu \nu} \rangle|_{\text{loc}}^{2} \approx \sigma^{2}(|E_{\mu} - E_{\nu}|), \tag{5}
\]

where \([ \cdots ]_{\text{loc}}\) denotes a local average over matrix elements corresponding to levels that are close to \(E_{\mu}\) and \(E_{\nu}\) in energy (see also Sec. V C for an explicit example). Put differently, the left-hand side in (5) is understood (and formally defined) analogously as when going over, e.g., from classical point particles to (local) particle densities, namely as the effective density of the perturbation’s squared matrix elements (in modulus, and “local” with respect to the spectrum of \(H_{0}\)). Accordingly, \(\sigma^{2}(E)\) in (5) is denoted as the perturbation profile, and is, by construction, a smooth \([22]\) and slowly varying function of \(E\) (compared to the mean level spacing \(\epsilon\)).

Semiclassical arguments \([23,24]\) as well as numerical evidence \([25–29]\) suggest that a rather common feature of realistic perturbations is a so-called banded structure of the perturbation matrix \(V_{\mu \nu}\) (see also Fig. 3 in Sec. V C below for a particular example). By definition, this means that the (coarse-grained) \(V_{\mu \nu}\) indeed depend only on \(E_{\mu} - E_{\nu}\) and that the perturbation profile \(\sigma^{2}(E)\) in (5) approaches zero for \(E \to \infty\). However, it should be emphasized that \(\sigma^{2}(E)\) is also admitted to remain finite for \(E \to \infty\); i.e., the matrix \(V_{\mu \nu}\) may but need not exhibit a banded structure \([30]\). Yet another common feature of many realistic perturbations is a so-called sparse matrix structure (large fraction of vanishing matrix elements \(V_{\mu \nu}\)), prominently arising, e.g., if the reference Hamiltonian \(H\) is noninteracting and \(V\) describes few-body interactions \([24,28,31,32]\). Again, our present approach is still compatible with a possibly (but not necessarily) sparse structure of \(V_{\mu \nu}\) [the local average in (5) then must extend over many nonvanishing matrix elements].

Our next goal is to establish the key role of the perturbation profile (5) for the deviations of the perturbed expectation values \(\langle A \rangle_{\mu \nu}(t)\) from the unperturbed \(\langle A \rangle_{\mu \nu}(t)\). The main idea is to consider not one particular \(V\), but rather an entire ensemble of perturbations, all of which share the property (5) with the “true” perturbation of interest, but are otherwise unbiased and rather arbitrary. More precisely, apart from the trivial constraint \(V_{\mu \nu}^{*} = V_{\nu \mu}\), we choose the matrix elements \(V_{\mu \nu}\) to be independent random variables following a probability distribution

\[
\rho_{\mu \nu}(v) := [\delta(V_{\mu \nu} - v)]_{\nu} = f_{E_{\mu} - E_{\nu}}(v), \tag{6}
\]

where \([ \cdots ]_{\nu}\) denotes the average over the ensemble of perturbations, and \([f(v)]_{\nu > 0}\) is a family of probability densities on \(\mathbb{R}\) or \(\mathbb{C}\) with mean zero and variance \(\sigma^{2}(E)\). Likewise, \(f_{0}(v)\) is a probability density on \(\mathbb{R}\) of vanishing mean and finite variance \([22]\). Note that the ensemble thus satisfies (5) in an ergodic sense, i.e., when replacing local averages \([ \cdots ]_{\text{loc}}\) by ensemble averages \([ \cdots ]_{\nu}\).

To arrive at a prediction for the perturbed dynamics \(\langle A \rangle_{\mu \nu}(t)\), we first evaluate the average behavior \([\langle A \rangle_{\mu \nu}(t)\] over all members of the considered ensemble of perturbations. Second, we consider the deviations \(\xi_{\mu}(t) := \langle A \rangle_{\mu}(t) - [\langle A \rangle_{\mu}(t)]_{\nu}\) for one particular realization from the average. It turns out \([17]\) that the variance \([\xi_{\mu}(t)]_{\nu}^{2}\) is inversely proportional to the number \(n_{\nu}\) of unperturbed levels mixed by the perturbation from assumption (iii). Exploiting (4), we can therefore conclude that, for the overwhelming majority of individual perturbations in the considered ensemble, the actual behavior \(\langle A \rangle_{\mu \nu}(t)\) is practically indistinguishable from the average \([\langle A \rangle_{\mu \nu}(t)\] so that the latter in fact correctly describes the dynamics under nearly all perturbations of the ensemble for sufficiently large system sizes. Results of this kind are also commonly known as typicality, concentration of measure, or ergodicity properties \([2–4]\).

Taking for granted that the perturbation profile (5) is indeed the essential quantity for deviations between the perturbed and unperturbed systems, we may expect that also the behavior of the true system of interest should follow the ensemble average. Unfortunately, it is hard to prove this for any given, concrete physical system. Nevertheless, a phenomenological justification by means of examples is possible for a variety of different models \([17,33]\); see also Sec. V below. For the rest, we observe that the probability distribution (6) is still rather arbitrary since we only fix the first two moments of the densities \(f_{E}(v)\). In principle and if available, additional information about the distribution of the true \(V_{\mu \nu}\) could thus be incorporated when choosing the \(f_{E}(v)\), but similarly as in the central limit theorem, these statistical properties turn out to be practically irrelevant, reinforcing the pivotal role of the second moment (5).

To conclude this section, we remark that the true perturbation will usually exhibit correlations (i.e., functional interdependencies) between the matrix elements \(V_{\mu \nu}\), which may arise, for example, due to the locality and few-body character of interactions \([34,35]\). Since such correlations are not accounted for in the considered perturbation ensembles, it is implicitly assumed that their effect on the dynamics is negligible. In practice, this particularly means that the reference Hamiltonian \(H\) should be sufficiently “clean” such that the individual terms constituting the perturbation \(V\) are in some sense “orthogonal” to those of \(H\). Notably, this rules out the possibility to reverse the roles by defining a new reference Hamiltonian \(H' := H_{0}\) and considering a perturbation \(\lambda V' := H - H' = -\lambda V\) to predict \(\langle A \rangle_{\mu \nu}(t)\) from \(\langle A \rangle_{\mu \nu}(t)\).

III. TYPICAL PERTURBED RELAXATION

Given the prominent role of the Hamiltonian as the generator of time evolution, it will be no surprise that the transformation matrices \(U_{\mu \nu}\) between the eigenbases of \(H\) and \(H_{0}\) [see Eq. (3)] are of particular importance to relate the unperturbed and perturbed dynamics. Especially relevant turns out to be the so-called overlap distribution \(u(E)\), which describes the squared magnitude of the \(U_{\mu \nu}\) averaged over the considered ensemble of perturbations,

\[
|U_{\mu \nu}(\lambda)|^{2} =: u(E_{\mu} - E_{\nu}). \tag{7}
\]

Because of the (approximate) constancy of the level density [assumptions (i) and (ii)] and the fact that the statistics of the \(V_{\mu \nu}\) in (6) only depend on \(E_{\mu} - E_{\nu}\), it follows that the statistics of the \(U_{\mu \nu}\) from (3) must be translationally invariant in energy,
and hence the second moment in (7) must only depend on the energy difference $E_m - E_v$.

Referring to Ref. [17] for the details, the typicality approach outlined below Eq. (6) then eventually yields that, for the overwhelming majority of perturbations in any admissible ensemble (6), the perturbed time evolution is given by

$$\langle A \rangle_{p(t)} = \langle A \rangle_p + \langle |g(t)|^2 \{ \langle A \rangle_{p(t)} - \langle A \rangle_p \} \rangle,$$  \hspace{1cm} (8)$$

We recall that $\langle A \rangle_{p(t)}$ is the reference dynamics observed under the unperturbed Hamiltonian $H$. Furthermore, the density operator $\tilde{\rho}$ appearing on the right-hand side of (8) is defined via its matrix elements $\langle \mu | \tilde{\rho} | \nu \rangle := \delta_{\mu \nu} \sum \bar{u}(E_n - E_z) \langle \nu | \rho(0) | \nu \rangle$, where $\bar{u}(E) := \int dE' D(E') u(E - E') u(E')$. In other words, $\tilde{\rho}$ may thus be viewed as the unperturbed diagonal ensemble associated with the initial state $\rho(0)$ which is in addition locally washed out via the function $\bar{u}(E)$, arising as the convolution of $u(E)$ with itself. According to Refs. [36–38], this operator $\tilde{\rho}$ can usually be well approximated by the microcanonical ensemble $\rho_{mc}$ corresponding to the pertinent energy window $I$ from (2). Finally, the so-called response profile $g(t)$ on the right-hand side of (8) is the Fourier transform of $u(E)$ from (7),

$$g(t) := \int dE D(E) u(E) e^{iEt}.$$  \hspace{1cm} (9)$$

In particular, it can be readily verified that $|g(0)|^2 = 1$ and $|g(t)|^2 \to 0$ as $t \to \infty$. According to (8), this function $g(t)$ thus describes how the unperturbed behavior is modified to approach the perturbed equilibrium value $\langle A \rangle_p$; i.e., it encodes the system’s response to the perturbation.

The remaining task is to compute the function $u(E)$ from (7). To this end, we introduce the resolvent $G(z) := (z - H_z)^{-1}$ of $H_z$, which encodes the overlaps on the left-hand side of (7) as $[U_{mv}]^2 \simeq \lim_{\eta \to 0^+} \langle \nu | \bar{G}(E_m - i\eta) - \bar{G}(E_m + i\eta) | \nu \rangle | \eta |^{2} 2 \pi i D(E_m) \rangle [1,39]$. Since the ensemble average of $G(z)$ can be written as $[G(z)]_\rho = G(z - H)$ with the scalar function $G(z)$ defined in a minute, we can exploit $D(E_m) \approx \varepsilon^{-1}$ [cf., assumptions (i) and (ii)] to arrive at

$$u(E) = \frac{\varepsilon}{\pi} \lim_{\eta \to 0^+} \text{Im} G(E - i\eta).$$  \hspace{1cm} (10)$$

Finally, the above introduced ensemble-averaged resolvent $G(z)$ itself can be obtained as the solution of the following nonlinear integral equation [17,24],

$$G(z) \left[ z - \lambda^2 \int dE D(E) G(z - E) \sigma^2(E) \right] = 1.$$  \hspace{1cm} (11)$$

In summary, the strategy to obtain a prediction for the perturbed relaxation thus is to follow the sequence of Eqs. (8)–(11) in reverse order: First, for a given perturbation profile $\sigma^2(E)$ and perturbation strength $\lambda$, we solve Eq. (11) for $G(z)$. Second, this gives us access to the overlap distribution $u(E)$ via Eq. (10). Third, evaluating its Fourier transform (9) we obtain the response profile $g(t)$, which then allows us, fourth, to predict $\langle A \rangle_{p(t)}$ from the unperturbed $\langle A \rangle_{p(0)}$ according to Eq. (8). Clearly, the first step, namely to solve the nonlinear integral equation (11), is the most demanding task. This problem is at the focus of the next section.

IV. EVALUATION OF THE ENSEMBLE-AVERAGED RESOLVENT

In this section, we will discuss solutions $G(z)$ of Eq. (11) and the resulting overlap distributions $u(E)$ from (10). We first consider in Sec. IV A two limiting cases for which analytical approximations will be obtained. Thereafter, we elaborate on how to solve Eq. (11) in the intermediate regime numerically using pseudospectral Chebyshev expansions [40,41]. The evaluation of the predicted dynamics and its comparison with explicit examples is deferred to the ensuing Sec. V.

A. Analytically tractable special cases

According to assumption (iv) from Sec. II, the perturbation profile $\sigma^2(E)$ from (5) is a well-behaving (continuous) function, so that the quantity

$$\tilde{\sigma} := \lim_{E \to 0^+} \sqrt{\sigma^2(E)}$$  \hspace{1cm} (12)$$

exists [22]. Essentially, $\tilde{\sigma}$ thus characterizes the “intrinsic strength” of the perturbations $V$.

As explained in Sec. II, the perturbation matrix $V_{\mu\nu}$ in the eigenbasis of the unperturbed Hamiltonian $H$ is often expected to exhibit a banded structure, meaning that its perturbation profile $\sigma^2(E)$ approaches zero as $E \to \infty$. The corresponding so-called band width or perturbation range may thus be quantified by

$$\Delta_v := \frac{1}{\tilde{\sigma}^2} \int_{0}^{\infty} dE \sigma^2(E).$$  \hspace{1cm} (13)$$

However, in full generality we will also admit cases where $\sigma^2(E)$ does not approach zero for large $E$. In such a case, but also when $\sigma^2(E)$ only decays very slowly with $E$, the band width $\Delta_v$ will be infinite.

Our first approximation starts from the observation that if the perturbation is sufficiently weak [sufficiently small $\lambda$ in (1)] then also the mixing of eigenvectors between the unperturbed and perturbed Hamiltonians should be weak in the sense that the concomitant eigenvector overlaps (7) are only non-negligible for small energy differences $E_m - E_v$ of the corresponding eigenvalues. In view of (10), we therefore inspect the case that the function $G(z - E)$ in the integrand in (11) exhibits (as a function of $E$, and for any preset $z$ of later relevance) a very narrow peak compared to variations of the perturbation profile $\sigma^2(E)$. Accordingly, the integral is dominated by the region around the maximum of $G(z - E)$ at $E \approx |z|$, and we can approximate $\sigma^2(E)$ by its central value $\sigma^2(|z|)$. Together with $D(E) \approx \varepsilon^{-1}$ [cf., assumption (i)] we thus obtain

$$G(z) = \frac{1}{z - \lambda^2 \sigma^2(|z|) C(z)/\varepsilon}$$  \hspace{1cm} (14)$$

with

$$C(z) := \int dE G(z - E).$$  \hspace{1cm} (15)$$

Exploiting once again that $G(z)$ exhibits a very narrow peak compared to the variations of $\sigma^2(|z|)$ implies with (12) that $\sigma^2(|z|) \approx \tilde{\sigma}^2$ for all the relevant values of $|z|$ for which $G(z)$ significantly deviates from zero. Furthermore, focusing in
view of (10) on arguments \( z = x - i\eta \) with \( x \in \mathbb{R} \), the quantity \( C(z) \) in (15) assumes the same constant value \( C(-i\eta) \) for all \( z \). In other words, \( G(z) \) in (14) can be written as \( 1/(z - c) \) for some constant \( c \in \mathbb{C} \). Consequently, when evaluated in the principal value sense, \( C(z) \) in (15) only depends on the sign of the imaginary part of the denominator in (14), yielding \( C(z) = \mp i\pi \) for \( \text{sgn(Im\,z)} = \pm 1 \) as the only consistent solution. Altogether, we thus arrive at the approximation

\[
G(z) = \frac{1}{z + i\text{sgn(Im\,z)}\Gamma/2} , \quad \Gamma := \frac{2\pi\lambda^2\sigma^2}{\varepsilon} ,
\]

and with (10) we conclude that \( u(E) \) approximately assumes the Breit-Wigner form

\[
u(E) = \frac{\varepsilon}{2\pi\varepsilon^2 + \Gamma^2/4} . \quad (18)
\]

Hence \( \Gamma \) quantifies the peak width of \( u(E) \), and likewise for \( G(z) \). Our initial assumption that \( G(z) \) is sharply peaked thus means that \( \sigma^2(E) \) must exhibit only small changes upon variations of \( E \) on the order of \( \Gamma \).Viewing the perturbation strength \( \lambda \) as variable and all other system properties as fixed, we may thus consider (16)–(18) as a weak perturbation approximation. Importantly, this approximation is expected to apply for practically any reasonable perturbation profile \( \sigma^2(E) \) provided the perturbation strengths \( \lambda \) are sufficiently small. In many cases, one furthermore expects that the band width (13) at the same time quantifies the scale on which \( \sigma^2(E) \) exhibits notable variations, yielding

\[
\Gamma \ll \Delta_v , \quad (19)
\]

as the pertinent condition for the validity of the above approximations. On the other hand, in cases where the variations of \( \sigma^2(E) \) remain relatively small for arbitrary \( E \), those approximations will actually apply to arbitrary coupling strengths \( \lambda \) (apart from the general restrictions in Sec. II).

Our second approximation is similar in spirit but complementary to the first one. Namely, we follow the same reasoning as before with the roles of \( G(z) \) and \( \sigma^2(E) \) reversed; i.e., we now consider the case that the perturbation profile \( \sigma^2(E) \) is sharply peaked compared to the variations of \( G(z) \) for arguments of the form \( z = E - i\eta \). In the integrand in (11), we thus approximate \( G(z - E) \approx G(z) \) and (as before) \( D(E) \approx \varepsilon^{-1} \), leading to

\[
\gamma^2G(z)^2/4 - zG(z) + 1 = 0 , \quad (20)
\]

\[
\gamma := \sqrt{8\Delta_v/\varepsilon\lambda \sigma} = \sqrt{4\Delta_v\Gamma/\pi} . \quad (21)
\]

Solving this algebraic equation for \( G(z) \) and observing that \( \text{sgn(Im\,G(z))} = -\text{sgn(Im\,z)} \) due to \( G(z) = [(z - \lambda V)^{-1}]_V \) [see above (10)], we obtain

\[
G(z) = \frac{2}{\gamma^2}[z - i\text{sgn(Im\,z)}\sqrt{\gamma^2 - z^2}] . \quad (22)
\]

Substituting into (10), we are left with the semicircular distribution

\[
u(E) = \frac{2\varepsilon}{\pi\gamma^2}\sqrt{\gamma^2 - E^2}\Theta(\gamma^2 - E^2) , \quad (23)
\]

where \( \Theta(x) \) denotes the Heaviside step function. The condition that \( \sigma^2(E) \) is sharply peaked thus means that \( G(z = E - i\eta) \) must exhibit only small changes upon variations of \( E \) on the order of \( \gamma \).Viewing the perturbation strength \( \lambda \) as variable and all other system properties as fixed, we may thus consider (21)–(23) as a strong perturbation approximation. More precisely, this approximation is expected to apply for practically any reasonable perturbation profile \( \sigma^2(E) \) provided the perturbation strengths \( \lambda \) are sufficiently large and provided that \( \sigma^2(E) \) does approach zero for large \( E \) in the first place. In particular, this is the case if the band width \( \Delta_v \) from (13) is finite. Furthermore, if \( \Delta_v \) at the same time quantifies the scale on which \( \sigma^2(E) \) exhibits notable variations, then the pertinent condition for the validity of the above approximations assumes the form

\[
\gamma \gg \Delta_v . \quad (24)
\]

Essentially, the overlap distribution \( u(E) \) from (7) is thus predicted to approximately assume the Breit-Wigner form (18) under the weak perturbation condition (19), and the semicircular form (23) under the strong perturbation condition (24), largely independently of any further details of the perturbation profile \( \sigma^2(E) \) from (5). In the intermediate regime, characterized by \( \Gamma \approx \gamma \), or equivalently

\[
\lambda \approx \lambda_c := \frac{\sqrt{2\pi\Delta_v}}{\pi \sigma} ,
\]

one thus expects a smooth crossover between these limiting cases (see also Fig. 1 below), which will depend on the detailed shape of the perturbation profile \( \sigma^2(E) \), and which in general will only be tractable by numerical means.

**B. Numerical treatment of the general case**

Our goal is to determine the overlap distribution \( u(E) \) according to (10) by numerically solving the nonlinear integral equation (11) for largely general perturbation profiles \( \sigma^2(E) \). As in the previous subsection, we thus can and will insist in (11) on arguments \( z = x - i\eta \) with \( x \in \mathbb{R} \) and \( \eta > 0 \) very small. As noted below (21), the relation \( G(x - i\eta) = [(x - i\eta - \lambda V)^{-1}]_V \) implies \( \text{Im}\,G(x - i\eta) \geq 0 \) for \( \eta > 0 \) and vice versa; i.e., the sign of the imaginary part of \( G(z) \) jumps when crossing the real line. For purely real \( z \), in turn, this implies that the solution of (11) becomes ambiguous, depending on whether one chooses to continue from the upper or lower half-plane. Bearing in mind that the latter option is appropriate in (10), we introduce the abbreviation

\[
G_+(x) := \lim_{\eta \to 0^+} G(x - i\eta) . \quad (26)
\]

Exploiting (as usual) that \( D(E) \approx \varepsilon^{-1} \) [cf., assumption (i)], the integral equation (11) can thus be rewritten for real-valued \( x \) as

\[
G_+(x)\left[ x - \lambda^2/\varepsilon \int dE\, G_+(x - E) \sigma^2(|E|) \right] = 1 . \quad (27)
\]
FIG. 1. Overlap distribution \( u(E) \) from (7) for three different perturbation profiles \( \sigma^2(E) \) as depicted in the insets of the left-most panel of each row, namely a step profile (33) in panel (a), an exponential profile (34) in panel (b), and a double-Breit-Wigner profile (35) with \( b_1 = 0.45, b_2 = 0.9, d = 3.5 \) in panel (c). In all three cases, we employed the same parameter values \( \epsilon = 1/512, \overline{\sigma}^2 = 0.2, \) and \( \Delta_v = 750 \epsilon = 1.46. \) In each row, the perturbation strength \( \lambda \) is increased from left to right as specified in the top right corner of each panel. Solid blue lines correspond to the numerical solution of (10) via (26)–(31), while dashed lines show the limiting Breit-Wigner (red [dark]) and semicircular (green [light]) distributions according to Eqs. (18) and (23), expected for weak and strong perturbations, respectively. As predicted below (25), the crossover between these two limits occurs around \( \lambda \approx \lambda_c \approx 0.05. \) Note that the vertical axes are scaled as indicated in the top left corner of each panel.

with the additional constraint that

\[
\text{Im} \, G_+(x) \geq 0. \tag{28}
\]

Our method of choice to solve Eq. (27) numerically is an expansion in terms of Chebyshev rational functions \( B_n(x) \) (\( n = 0, 1, \ldots \)), which are derived from the Chebyshev polynomials of the first kind \( T_n(x) \) by compacting the real line,

\[
B_n(x) := T_n \left( \frac{x}{\sqrt{x^2 + \ell^2}} \right). \tag{29}
\]

Here, \( \ell \) is an arbitrary, fixed parameter that sets the scale for compactification and should roughly reflect the typical scale of the function to be expanded for optimal convergence [41]. Hence, we express

\[
G_+(x) = G^R_+(x) + iG^I_+(x), \tag{30}
\]

where the real-valued functions \( G^R_+(x) \) and \( G^I_+(x) \) are truncated Chebyshev series, i.e.,

\[
G^R_+(x) := \sum_{n=0}^{M} G^R_n B_n(x) \quad \text{and} \quad G^I_+(x) := \sum_{n=0}^{M} G^I_n B_n(x). \tag{31}
\]

The (real-valued) coefficients \( G^R_n \) and \( G^I_n \) are then to be determined such that \( G_+(x) \) from (30) satisfies (27) and (28) as well as possible. For given expansion coefficients \( G := (G^R_0, G^I_0, \ldots, G^R_M, G^I_M) \) and \( x \), the residual [i.e., the violation of Eq. (11)] is defined as

\[
R(G, x) := G_+(x) \left[ x - \frac{\lambda^2}{\epsilon} \int dE \, G_+(x - E) \sigma^2(|E|) \right] - 1 \tag{32}
\]

with \( G_+(x) \) from (30) and (31). We minimize \( |R(G, x)|^2 \) by means of pseudospectral methods [40,41], requiring \( \text{Re} \, R(G, x_m) = \text{Im} \, R(G, x_m) = 0 \) for a discrete set of real-valued collocation points \( x_m (m = 0, 1, \ldots, M). \) A common choice for these \( x_m \) is to use the roots of the \( (M + 1) \)th Chebyshev rational function \( B_{M+1}(x) \), so that the pseudospectral method coincides with a spectral expansion when an optimal Gaussian quadrature rule is used to calculate inner products numerically [40,41].

Altogether, forcing \( \text{Re} \, R(G, x_m) = \text{Im} \, R(G, x_m) = 0 \) results in a set of \( 2(M + 1) \) algebraic equations for the \( 2(M + 1) \)
unknown expansion coefficients \( G_{\nu}^E, \ G_{\nu}^E \in \mathbb{R} \). This system of
equations is then solved iteratively by the Newton-Raphson
method using either of the limiting distributions (18) or (23)
for the first initial guess, and gradually varying \( \lambda \) across the in-
termediate regime thereafter. If the initial guess is sufficiently
close to the actual solution and satisfies \( \text{Im} G_{\nu}(x) \geq 0 \), this
ensures that also the finally obtained approximation will fulfill
the constraint (28).

In Fig. 1, we display the so-obtained numerical solutions \( u(E) \) in (10) for different perturbation profiles \( \sigma^2(E) \) and various perturbations strengths \( \lambda \) along with the limiting
Breit-Wigner functions (18) expected for small \( \lambda \) and the
semicircular functions (23) expected for large \( \lambda \). The selected
perturbation profiles consist of a step function,
\[
\sigma^2(E) = \bar{\sigma}^2 \Theta(\Delta_n - E), \tag{33}
\]
an exponential function,
\[
\sigma^2(E) = \bar{\sigma}^2 e^{-E/\Delta_n}, \tag{34}
\]
and a double-Breit-Wigner function,
\[
\sigma^2(E) = \bar{\sigma}^2 \frac{b_1^2 + d^2}{(b_1^2 + E^2)(b_2^2 + (E - d)^2)}. \tag{35}
\]
All three perturbation profiles are also shown in the insets of the left panels in Fig. 1. Parameters are chosen such that in all cases \( \varepsilon = 1/512 \) (mean level spacing), \( \bar{\sigma}^2 = 0.2 \) [cf., Eq. (12)], and \( \Delta_n = 750 \varepsilon = 1.46 \) [bandwidth, cf., Eq. (13)],
yielding a value of \( \lambda_c \approx 0.05 \) for the crossover coupling
strength in (25). Moreover, the order of the Chebyshev ex-
pansions is \( M = 80 \) throughout, with the parameter \( \ell \) varying between 0.5 and 8 [roughly optimizing the global residual (32)].

For each of the three profiles (33)–(35), the predicted
crossover from the Breit-Wigner to the semicircular shape of
\( u(E) \) is clearly visible as \( \lambda \) is increased. The intermediate
regime, where neither the Breit-Wigner nor the semicircular
distribution offers a satisfactory approximation, appears to be somewhat smaller for the discontinuous step profile
than for the smooth exponential and double Breit-Wigner
profiles. In any case, in this intermediate regime there is a
(relatively mild) dependence of \( u(E) \) on the detailed shape
of \( \sigma^2(E) \). It therefore seems reasonable to expect that—at
least in principle—it may be possible to reconstruct from a
sufficiently precisely known function \( u(E) \) the underlying
perturbation profile \( \sigma^2(E) \).

V. EVALUATION OF THE RELAXATION DYNAMICS
AND EXAMPLES

With our above obtained results for the overlap distribution
\( u(E) \) at hand, we now turn to their implications for the
response profile \( g(t) \), which governs the deviations of the
perturbed from the unperturbed relaxation behavior according
to (8). Specifically, we will first address in Sec. VA some
more general issues, while in the subsequent Secs. VB and
VC, we will compare our theoretical prediction (8) with
two explicit examples of random-matrix and spin models,
respectively.

A. Response profile

Exploiting in (9) our usual approximation \( D(E) \approx \varepsilon^{-1}
\) [cf., assumption (i)], the response profile \( g(t) \) can be readily
obtained via Fourier transformation from our analytical and
numerical findings for \( u(E) \) in the previous Sec. IV. For the
two analytically tractable special cases from Sec. IV A, the
Fourier transformation can again be performed analytically,
whereas for the numerical solutions from Sec. IV B, also the
Fourier transformation is only possible by numerical means.

In the limit of weak perturbations, when \( u(E) \) assumes the
Breit-Wigner form (18), one readily finds along these lines that
\( g(t) \) amounts to an exponential decay,
\[
g(t) = e^{-\Gamma |\gamma t|/2}, \tag{36}
\]
where the rate \( \Gamma \) is the full width at half maximum of \( u(E) \) as
defined in (17).

Likewise, for (moderately) strong perturbations such that
\( u(E) \) takes the semicircular shape (23), its Fourier transform
is
\[
g(t) = \frac{2 J_1(\gamma t)}{\gamma t}, \tag{37}
\]
where \( J_1(x) \) is the Bessel function of the first kind of order 1,
and \( \gamma \) as specified in Eq. (21) is the radius of the semicircle.

In the intermediate regime, our findings for \( u(E) \) imply that
\( g(t) \) must exhibit a crossover between these two limiting
behaviors. Calculating the Fourier transforms of the numerical
solutions for \( u(E) \) from Fig. 1, we obtain the solid curves
shown in Fig. 2 for \(|g(t)|^2\), which is the actually relevant
quantity in (8). This illustrates quantitatively the expected
crossover from (36) to (37) with increasing \( \lambda \).

The first general conclusion is that the perturbed relaxation
becomes faster with increasing \( \lambda \). Quite obviously, the under-
lying physical reason is a corresponding broadening of \( u(E) \nwith increasing \( \lambda \), which in turn indicates (as expected) that
an increased number of unperturbed energy levels are coupled
by the perturbation according to (7).

The second general conclusion is that the functions \( g(t) \)
become independent of any further details of the perturbation
profile \( \sigma^2(E) \) for asymptotically large or small \( \lambda \), while some
(rather moderate) functional dependence on \( \sigma^2(E) \) remains
in the intermediate regime. Again, the underlying reasons are our
analogous observations for the overlap distributions \( u(E) \) in
the preceding section. Though the functional dependence of
\( g(t) \), and thus of the perturbed relaxation in (8), is quantita-
tively rather weak, it still may be possible, at least in principle,
to infer the (coarse-grained) perturbation profile (5) of the
specific perturbation \( V \) for some given many-body system (1)
from the observable temporal relaxation via (8).

B. Random matrix example

To verify that the theoretical prediction (8) indeed de-
scribes the behavior of many-body quantum systems [pro-
vided that assumptions (i) through (iv) from Sec. II hold], we
finally compare it to explicit numerical examples.

The first example is a (in some sense artificial) random
matrix model that satisfies the requirements from Sec. II by
construction and thus serves as a test bed for the validity of
the approximations employed in the derivation of Eq. (8) (see
also Ref. [17]). The reference Hamiltonian has equally spaced energy levels \( E_\nu = v \nu \) with \( \nu = 1/512 \). The perturbation \( V \) is a complex Hermitian random matrix distributed according to (6) with

\[
f_E(v) = (1 - p) \delta(v) + \frac{p e^{-|v|^2/\sigma^2(E)}}{\pi \sigma^2(E)} \quad (E > 0).
\]

On average, the matrices \( V_{\mu \nu} \) are thus sparse with a fraction \( p \) of nonvanishing entries following a complex normal distribution of variance \( \sigma^2(E) \) for \( \mu < \nu \), and \( V_{\mu \mu} = V_{\nu \nu} \). For simplicity, the diagonal matrix elements \( V_{\nu \nu} \) are sampled similarly, but with a real normal distribution for the nonvanishing entries. Consequently, the perturbation profile (5) is given by

\[
\sigma^2(E) = p \tilde{\sigma}^2(E). 
\]

Specifically, we implemented the three perturbation profiles (33)–(35) with \( \tilde{\sigma}^2 = p = 0.2 \) and \( \Delta_v = 1.46 \) (corresponding to about 750 levels).

The initial state \( \rho(0) = \ket{v_0} \bra{v_0} \) is an eigenstate of the reference Hamiltonian \( H \) from the middle of the spectrum, and we observe its survival probability or fidelity [42,43], i.e., \( A = \rho(0) \). Hence \( \bra{A} \rho(t) \ket{A} = 1 \) for all \( t \) while \( \bra{A} \rho \ket{A} \approx 0 \) for a sufficiently large energy window \( I \) from (2), so that the prediction (8) reduces to

\[
\bra{A} \rho(t) \ket{A} = |g(t)|^2. 
\]

In other words, recording the dynamics in this setup for one particular perturbation sampled from (38), we should exactly recover the solid curves in Fig. 2. The dashed lines in the figure represent one such example dynamics for a Hilbert space of dimension \( 2^{14} = 16,384 \) and an initial eigenstate \( \ket{v_0} \) with \( v_0 = 2^{13} = 8192 \).

The main conclusion is that the simulation results indeed agree almost perfectly with the theoretically predicted solid curves throughout the entire crossover regime.

### C. Spin lattice example

Finally, we test the theoretical prediction (8) in a more realistic two-dimensional spin-\( \frac{1}{2} \) model. We consider a square lattice of \( L \times L \) sites as sketched in Fig. 3(a), where the reference Hamiltonian couples nearest neighbors with an isotropic spin-spin interaction,

\[
H = J \sum_{i,j=1}^{L-1} \sigma_{i,j} \cdot (\sigma_{i+1,j} + \sigma_{i,j+1}).
\]

Here \( \sigma_{i,j} := (\sigma_{i,j}^x, \sigma_{i,j}^y, \sigma_{i,j}^z) \) with \( \sigma_{\alpha}^\mu \) denoting the Pauli matrices acting on site \((i,j)\). The perturbation adds spin-flip terms between next nearest neighbors,

\[
V = \sum_{i,j=1}^{L-1} \sum_{\alpha=x,y} (\sigma_{i,j}^\alpha \sigma_{i+1,j+1}^\alpha + \sigma_{i+1,j}^\alpha \sigma_{i,j+1}^\alpha). 
\]

In all of the numerics presented here, we used \( L = 4 \) and \( J = 1 \), and we focused on the sector with vanishing total magnetization in the \( z \) direction.

To obtain the perturbation profile (5) of \( V \), we first fix an energy window \( I \) by choosing the central 60\% of energy levels, which comprise a total of 7722 states ranging from \( E = -8.8 \) to \( E = 5.8 \), implying a mean level spacing \( \epsilon = 0.0019 \). Next, we compute the matrix elements \( V_{\mu \nu} \) with \( E_{\mu}, E_{\nu} \in I \) by diagonalizing the reference Hamiltonian \( H \). A coarse-grained view of the resulting matrix is shown in Fig. 3(b), visualizing the bandedness of the perturbation matrix. We proceed by binning the \( V_{\mu \nu} \) according to the energy difference \( E_{\mu} - E_{\nu} \) of the associated levels and evaluate the average of \( |V_{\mu \nu}|^2 \) within each bin. The obtained relation between the coarse-grained \( |V_{\mu \nu}|^2 \) and \( E_{\mu} - E_{\nu} \) is displayed as a black curve in Fig. 3(c), indicating an approximately exponential dependency. The function \( \sigma^2(E) \) is then determined by fitting the exponential form (34) to the empirical distribution, yielding the red line in Fig. 3(c) with \( \sigma^2 = 0.00502 \) and \( \Delta_v = 7.32 \). This implies a value of \( \lambda_c = 0.75 \) for the predicted location of the crossover (25) between the exponential and Bessel-type decay characteristics (36) and (37), respectively.

As a first observable, we investigate the magnetization correlation \( m_z \) in the \( z \) direction between next-nearest neighbors from the center of the lattice,

\[
m_z := \sigma_{z,2}^z \sigma_{z,3}^z. 
\]

One could consider these two spins at \((2,2)\) and \((3,3)\) as the system and all other surrounding spins as a bath. In the reference Hamiltonian \( H \), the system spins can thus only interact via the bath, whereas the perturbation \( V \) adds a direct interaction between them.
Hamiltonian (41) in a central energy window and $\Delta_1$ dashed links correspond to those coupled by the perturbation $V$. Highlighted in red are the sites (2,2) and (3,3), on which the two considered observables [see Eqs. (43) and (46)] are supported. (b) Squared matrix elements $|V_{ij}|^2$ of the perturbation (42) in the eigenbasis of the reference Hamiltonian (41) in a central energy window $I$ of 7722 states (60% of the total) in the zero-magnetization sector, averaged over blocks of 100 $\times$ 100 levels. (c) Coarse-grained perturbation profile (5) (black, bin width 0.01) and fit to the exponential form (34) with $\hat{\sigma}^2 = 0.00502$ and $\Delta_1 = 7.32$ (red). The inset shows the same data with a logarithmically scaled y axis.

For the initial state $\rho(0) = |\psi\rangle\langle\psi|$, we choose those two system spins at (2,2) and (3,3) to be in the up state, while the bath is supposed to be at equilibrium, which we emulate by choosing a Haar-distributed random vector in the bath’s subspace. However, to ensure assumption (i) of a well-defined macroscopic energy, we finally apply a Gaussian projection $\Pi_{E,\Delta_\varepsilon}$ of mean energy $E = 0$ and standard deviation $\Delta_\varepsilon = 2$ to the so-obtained state, simulating a macroscopic measurement of the system energy that yielded $E = 0$ [44–46]. If $|\psi\rangle$ denotes a Haar-distributed random vector on the full (zero-magnetization) Hilbert space, we thus have

$$|\psi\rangle \propto \Pi_{E,\Delta_\varepsilon}\sigma_0^+\sigma_3^+|\phi\rangle$$

with $\sigma_{i,j}^\pm := \sigma_{i,j}^x \pm i\sigma_{i,j}^y$ and

$$\Pi_{E,\Delta_\varepsilon} \propto \sum_v e^{-(E_v - E)^2/2\Delta_\varepsilon^2} |v\rangle\langle v|$$

In Fig. 4(a), we compare the observed dynamics obtained by exact diagonalization (dashed lines) with our theoretical prediction (8) (solid lines) for several perturbation strengths $\lambda$. For the theoretical prediction, we use the numerical reference dynamics (i.e., the dash-dotted black curve with $\lambda = 0$) for $(m_\kappa)_{\mu(\nu)}$. The function $g(t)$ is the Fourier transform of $u(E)$ calculated as explained in Sec. IVB from the empirically determined approximate perturbation profile $\sigma^2(E)$, i.e., the red curve in Fig. 3(c). The so-obtained response profiles $g(t)$ are also displayed in the inset of Fig. 4(a). Since the long-time limiting values exhibit some finite-size variations, we do not use the microcanonical value $(m_\kappa)_{\mu(\nu)} = -0.0805$ (within the 60% window $I$) for $(m_\kappa)_{\mu}$, but instead compute the predicted coarse-grained diagonal ensemble $\hat{\rho}$ directly as detailed below Eq. (8), making use of our solution for $u(E)$ and the known occupations $\langle v|\rho(0)|v\rangle$ of the initial state from (44). The resulting quantitative values of $(m_\kappa)_{\mu}$ for the various perturbations strengths $\lambda$ are given in the figure caption.

The agreement between theory and numerics is very good despite the rather small system size and several idealizations. In particular, the assumptions of a homogeneous density of states [assumption (i)], of an exponential perturbation profile [assumption (iv) and Fig. 3(c)], and of uncorrelated matrix elements $V_{\mu\nu}$ [see above (6)] are all violated to some extent and are thus potential origins of the visible small deviations in Fig. 4 for short times. The fluctuations for longer times, in contrast, are likely caused predominantly by finite-size effects. We emphasize that there are no free parameters in the theoretical prediction; all ingredients in (8) were extracted directly from properties of the model (41) and (42).

As a second observable, we consider the spin-flip or hopping correlation $j_c$ between the same sites (2,2) and (3,3) from the center of the lattice in Fig. 3(a),

$$j_c := \sigma_2^x\sigma_3^y - \sigma_2^y\sigma_3^x = \frac{1}{2i}(\sigma_2^x\sigma_3^+ - \sigma_2^+\sigma_3^x).$$

where $\sigma_{i,j}^\pm := \sigma_{i,j}^x \pm i\sigma_{i,j}^y$. For the initial state, we employ a dynamical typicality setup [47,48] to prepare the system far from equilibrium, choosing

$$|\psi\rangle \propto \Pi(1 + \kappa j_c)\Pi|\phi\rangle,$$

where $|\phi\rangle$ is a Haar-distributed random state as before, $\Pi$ is a projector onto the central 2048 states in the zero-magnetization sector [ensuring assumption (i)], and $\kappa$ is a real parameter (in the examples, we use $\kappa = 2$).

A similar comparison as for $m_\kappa$ between numerical simulations and the theoretical prediction (8) is shown for the hopping correlation $j_c$ from (46) in Fig. 4(b). In particular, the functions $g(t)$ are the same in both panels of Fig. 4. On the other hand, in this setup $(j_c)_{\mu(\nu)}$ is well approximated by the thermal expectation value $(j_c)_{\mu(\nu)} = 0$ (by symmetry), so that we used this value throughout. Altogether, this amounts again to an entirely parameter-free prediction of the perturbed dynamics, which agrees well with the actually observed behavior.
VI. CONCLUSIONS

We investigated the response of quantum many-body systems to weak-to-moderate perturbations within a nonperturbative typicality framework. In particular, we presented a method to theoretically predict time-dependent expectation values of observables for the unperturbed system from the unperturbed relaxation behavior. This prediction (8) entails that the perturbed relaxation resembles the unperturbed one but is modified by a characteristic response profile function $g(t)$ that pushes the system toward a coarse-grained diagonal ensemble state, which can usually be identified with the pertinent thermal state. The function $g(t)$, in turn, is essentially determined by the perturbation profile, i.e., the locally averaged squared absolute value (5) of the perturbation’s matrix elements $V_{\mu\nu}$, in the unperturbed basis.

For asymptotically weak perturbations, the response profile $g(t)$ describes an exponential decay, where the decay rate corresponds to the energy scale across which the perturbation mixes unperturbed eigenstates, scaling quadratically with the perturbation strength $\lambda$. Broadly speaking, this may be understood as a nonperturbative justification of Fermi’s golden rule in a many-body setting.

The nonperturbative character of our method becomes manifest as the perturbation strength is increased. Our results then predict a crossover of $g(t)$ toward the Bessel-type shape (37), whose inverse relaxation timescale $\gamma$ still quantifies the mixing of energy levels, but now scales linearly with $\lambda$ and additionally depends on the energy range $\Delta_E$ of the perturbation.

We verified all those theoretical predictions in an explicit example of a spin system on a $4 \times 4$ square lattice. Using exact diagonalization to determine the perturbation profile of the applied perturbation empirically [cf., Fig. 3(c)], the function $g(t)$ derived from it indeed describes the actually observed perturbed dynamics remarkably well as long as the key assumptions (i) through (iv) collected in Sec. II are satisfied. Notably, the theory does not involve any free parameters; i.e., all quantities were determined first hand from the underlying spin model. Since the perturbation profile is the only variable input for the theory, this establishes that said profile encodes the dynamical response on a fundamental level.

Then again, the correspondence between the perturbation profile and the dynamical response may in principle be exploited the other way round, too. The rapidly improving experimental capabilities to observe time-dependent expectation values of mesoscopic quantum systems may thus offer a way to probe the (coarse-grained) matrix elements of applied perturbations. A similar proposal to extract matrix structures from dynamics can also be found in the recent work [49] using periodic driving and working in the regime of weak perturbations governed by the exponential law (36). Our present approach can be considered complementary in that it avoids time-dependent manipulations and extends to significantly stronger perturbations. Given the important role of matrix elements in the energy eigenbasis for the dynamics in general and for questions of equilibration and thermalization (e.g., the eigenstate thermalization hypothesis) in particular, this sets up new possibilities to explore the underlying mechanisms by means of time series analysis.

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In view of the sentence around Eq. (3), the initial state may typically populate a wider range of levels in the perturbed system than in the unperturbed one, i.e., the width of the distribution of occupied levels may increase by an amount on the order of $\Gamma$. As detailed in Ref. \[17\], this implies that the width $\Delta$ of the energy window $I$ introduced above Eq. (2) must be sufficiently large and $\Gamma$ must remain sufficiently small so that $\Gamma \ll \Delta$.

We note that this assumption is somewhat similar to (but still considerably weaker than) the (off-diagonal) eigenstate thermalization hypothesis (ETH) \[2,36, 50, 51\].

The variance of the diagonal elements $V_n$ need not be of the same order as that of the close-by off-diagonal elements; i.e., $\sigma^2(E)$ from (5) need not be continuous as $E \to 0$ \[52\].

The variance of the off-diagonal elements $V_n$ need not be of the same order as that of the close-by off-diagonal elements; i.e., $\sigma^2(E)$ from (5) need not be continuous as $E \to 0$ \[52\].

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